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ON DUAL LOCALLY UNIFORMLY ROTUND NORMS

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ABSTRACT

We show that the existence of an equivalent dual LUR norm on a dual Banach space can be characterized by a topological property similar to the fragmentability. The compact spaces homeomorphic to weak* compact subsets of a dual LUR Banach space have the same properties as the class of Radon–Nikodým compact spaces.

1. Introduction

A Banach space X is said to be Asplund if every convex function on X is Fréchet differentiable on a dense $\mathcal{G}_\delta$-set. If a Banach space has an equivalent Fréchet differentiable norm then it is Asplund, but the converse is not true; see [2], for example. The Šmulyan criterion provides a method to construct an equivalent Fréchet differentiable norm on X: any equivalent norm on X is Fréchet differentiable provided that its dual norm on X^* is locally uniformly rotund (the notation used in this paper about Banach spaces is standard and it can be found in most of the books; see [2], for instance).

Definition 1.1: Let X be a Banach space endowed with a norm $\|.\|$ and let S_X denote its unit sphere. The norm $\|.\|$ is said to be locally uniformly rotund (LUR), if $\lim_k \|x-x_k\| = 0$ whenever $x, x_k \in S_X$ are such that $\lim_k \|x+x_k\| = 2$.

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In this paper we study how close is the property of being the dual of an Asplund space to having an equivalent dual LUR norm. We shall need the following topological definitions. The first one has been introduced by Jayne and Rogers in [12].

Definition 1.2: Let (X,τ) be a topological space and let d be a metric on X. We say that X

(1) is fragmentable by d if for every $\varepsilon > 0$ and every nonempty $A \subset X$ there is $U \in \tau$ such that $A \cap U \neq \emptyset$ and $\operatorname{diam}(A \cap U) < \varepsilon$;
(2) has property $P(d,\tau)$ if there is a sequence (A_n) of subsets of X, such that for every $x \in X$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $\operatorname{diam}(A_n \cap U) < \varepsilon$.

Namioka and Phelps showed in [17] that a Banach space X is Asplund if and only if the unit ball of X^* endowed with the weak* topology is fragmented by the norm. They also showed [17] that if a dual Banach space X^* has an equivalent w^*-Kadec norm, that is, the weak* and the norm topologies agree on the unit sphere, then X is Asplund. Property P was introduced in [18] for pairs of topologies, but when stated as above it is equivalent to properties introduced and studied by Hansell [7] and Jayne, Namioka and Rogers [10]. The main result of this work is the following theorem which says that dual LUR renormability of a dual space X^* is a nonlinear topological property.

THEOREM 1.3: *Let X^* be a dual Banach space. The following conditions are equivalent:*

(i) *X^* admits an equivalent dual LUR norm.*
(ii) *X^* admits an equivalent w^*-Kadec norm.*
(iii) *X^* has $P(\|.\|,w^*)$.*

Statement (iii) above completes the characterizations of renormability given in [19]. Let us mention that there are no analogous results in Banach spaces for the weak topology. There exists a Banach space having a Kadec norm but with no equivalent strictly convex norm [8]. It is unknown whether every σ-fragmentable Banach space (in particular, if X has $P(\|.\|,w)$) has an equivalent Kadec norm [10].

We prove an interpolation result in the spirit of the results by Davis, Figiel, Johnson and Pelczyński for Eberlein compacta [4] and Namioka for Radon–Nikodým compacta [4, 16]. It can also be regarded as a "reciproque" of the transfer technique of Godefroy, Troyanski, Whitfield and Zizler for LUR renorming [6, 2].

THEOREM 1.4: *Let X be a Banach space, and let $K \subset X^*$ be a w^*-compact subset which has $P(\|.\|, w^*)$. Then there exists a Banach space Y such that Y^* has a dual LUR norm and a bounded linear operator $T\colon X \to Y$ with dense range such that $K \subset T^*(B_{Y^*})$.*

A compact Hausdorff space is said to be a Radon–Nikodým compact if it is homeomorphic to a weak*-compact subset of a dual Banach space having the Radon–Nikodým property. A result of Namioka states that a weak*-compact subset of a dual Banach space X^* which is fragmented by the norm of X^* is a Radon–Nikodým compact. All these facts suggest that we introduce the following class of compact Hausdorff spaces.

Definition 1.5: A compact Hausdorff space K is called a **Namioka–Phelps compact** if it is homeomorphic to a weak*-compact subset of a dual Banach space having a dual LUR norm.

Clearly, any Namioka–Phelps compact space is Radon–Nikodým. Namioka characterizes internally the Radon–Nikodým compacta as those compact Hausdorff spaces which are fragmented by a lower semicontinuous metric. We will prove an analogous result.

THEOREM 1.6: *A compact Hausdorff space K is Namioka–Phelps if and only if it has property $P(d,\tau)$ with some τ-lower semicontinuous metric d.*

If K is a Radon–Nikodým compact, then the space $C(K)$ is weak-Asplund, that is, every convex function on $C(K)$ is Gâteaux differentiable on a dense $\mathcal{G}_\delta$-set. Similarly, we obtain the following result.

THEOREM 1.7: *If K is a Namioka–Phelps compact space, then $C(K)$ has an equivalent Gâteaux differentiable norm.*

The organization of the next sections is as follows. In section 2 we study compact spaces having the property P with some metric showing the analogue with the properties of fragmentable compact spaces studied by Namioka in [16]. In section 3 we prove the main result of this paper concerning the characterization of the existence of equivalent dual LUR norms in a dual Banach space. Finally, we study embedding properties of the Namioka–Phelps compact spaces.

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2. Compact spaces with property P

A network of some topology is a family of subsets such that any open set is a union of subsets from that family. In [18] we introduced the property P for a couple of topologies. If X is a set and δ and τ topologies on X, we say that X has $P(\delta,\tau)$ if there is a sequence (A_n) of subsets of X such that for every $x \in X$ and every $V \in \delta$ with $x \in V$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U \subset V$. This property can be reformulated in terms of networks as follows: X has $P(\delta,\tau)$ if $\{A_n \cap U : n \in \mathbb{N}, U \in \tau\}$ is a network for δ. One can observe that this definition of property P extends Definition 1.2. We say that a topological space X has property $P(\delta,\tau)$ with τ-closed sets, if the sets $A_n \subset X$ can be taken τ-closed. The following is in [18].

Lemma 2.1: *Suposse that a set X has $P(d,\tau)$ with a sequence of subsets (A_n). If the metric d is τ-lower semicontinuous, then X has $P(d,\tau)$ with the sequence $(\overline{A_n}^\tau)$. In particular, X has $P(d,\tau)$ with τ-closed sets.*

Proposition 2.2: *Let X be a set, δ and τ two topologies on X. The following statements are equivalent:*

(i) *X has $P(\delta,\tau)$ with τ-closed sets.*

(ii) *There is a τ-lower semicontinuous function $F: X \to \mathbb{R}$ such that for every net (x_ω) with τ-$\lim_\omega x_\omega = x$ and $\lim_\omega F(x_\omega) = F(x)$, then δ-$\lim_\omega x_\omega = x$.*

A real function with the property stated in (ii) *will be called a Kadec function.*

Proof: (ii) $\Rightarrow$ (i) For every $x \in X$ and every $V \in \delta$ with $x \in V$ there is $U \in \tau$ and $\varepsilon > 0$ such that if $y \in U$ and $|F(y) - F(x)| < \varepsilon$, then $y \in V$. Let (r_n) be an enumeration of the rational numbers. Define

$$A_n = \{y \in X : F(y) \le r_n\}.$$

The sets A_n are τ-closed because F is τ-lsc. We claim that X has $P(\delta,\tau)$ with the sequence A_n. Indeed, take rationals $r_m < F(x) < r_n$ and $r_n - r_m < \varepsilon$. Consider the τ-open set $U' = U \setminus A_m$. Then we have that

$$x \in A_n \cap U' \subset V,$$

which proves the claim.

(i) $\Rightarrow$ (ii) Let Ξ_A be the characteristic function of the set A. Consider the series

$$F(x) = \sum_{n=1}^{\infty} 4^{-n} \Xi_{X \setminus A_n}(x).$$

It follows that F is τ-lsc. Let (x_ω) be a net with $\tau\text{-}\lim_\omega x_\omega = x$ and $\lim_\omega F(x_\omega) = F(x)$. We claim that $\delta\text{-}\lim_\omega x_\omega = x$. Indeed, a simple reasoning gives us that

$$\lim_\omega \Xi_{X\setminus A_n}(x_\omega) = \Xi_{X\setminus A_n}(x)$$

for every $n \in \mathbb{N}$. Now, for every δ-neighbourhood V of x there is n and $U \in \tau$ such that $x \in A_n \cap U \subset V$. Since $\Xi_{X\setminus A_n}(x_\omega)$ must be constant for ω big enough, we deduce that $x_\omega \in A_n$. Also, for ω big enough, $x_\omega \in U$. Thus $x_\omega \in V$. This shows the δ-convergence of (x_ω) to x. ∎

The following result compares with [16, Theorems 1.2, 1.4].

COROLLARY 2.3: *Every weak compact subset of a Banach space has $P(\|.\|, w)$, and every Eberlein compact space is Namioka-Phelps.*

Proof: Without loss of generality, we assume that $X = \overline{\text{span}}^{\|\cdot\|}(K)$. Then the space X will have an equivalent LUR norm $|||.|||$, which is in particular a Kadec norm. Then apply Proposition 2.2. Any Eberlein compact space is isomorphic to a weak compact subset of a reflexive space, which has an equivalent LUR norm which clearly is dual. ∎

A family of subsets of a topological space is said to be isolated if every point belonging to a subset of the family has a neighbourhood that does not meet another member of the family. A family of subsets is said to be σ-isolated if it is the union of a countable number of isolated families. Hansell studied in [7] the class of topological spaces having a σ-isolated network as a natural generalization of metrizable spaces; see also [15]. The following result is a consequence of the work of Hansell, and it shows the relation between fragmentability and property P.

THEOREM 2.4: *Let (K,τ) be a compact Hausdorff space and let d be a τ-lower semicontinuous metric on K. The following are equivalent:*

(i) *K has $P(d,\tau)$.*
(ii) *d has a network which is σ-isolated with respect to τ.*
(iii) *τ has a σ-isolated network and d fragments (K,τ).*

Proof: (i) ⇒ (iii) Any τ-lsc metric on K is finer than τ. Let $\mathfrak{B} = \bigcup_{m=1}^\infty \mathfrak{B}_m$ be a basis of d such that every family $\mathfrak{B}_m$ is d-discrete. Fix $n, m \in \mathbb{N}$ and $E \in \mathfrak{B}_m$. Define

$$H_E = \{x \in A_n \colon \exists U \in \tau \text{ s.t. } x \in A_n \cap U \subset E\}.$$

It is easy to see that $\mathfrak{N}_{n,m} = \{H_E: E \in \mathfrak{B}_m\}$ is τ-isolated and $\mathfrak{N} = \bigcup_{n,m} \mathfrak{N}_{n,m}$ is a network of d. Since τ is coarser than d, we have that $\mathfrak{N}$ is a σ-isolated network of τ. On the other hand, it is shown in [18] that if K has $P(d,\tau)$, then (K,τ) is σ-fragmented by d. Since d is τ-lsc and (K,τ) is compact, a result from [11] states that (K,τ) is fragmented by d.

(iii) $\Rightarrow$ (ii) If (K,τ) has a σ-isolated network, then it is in particular hereditarily weakly θ-refinable, that is, every family of open sets in X has a σ-isolated (not necessary open) refinement. Hansell shows [7] that if a hereditarily weakly θ-refinable space is fragmented (or σ-fragmented) by some metric d, then the topology of d has a network $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$ such that every $\mathfrak{N}_n$ is σ-isolated respect to τ.

(ii) $\Rightarrow$ (i) If $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$ is a network of d such that every $\mathfrak{N}_n$ is σ-isolated with respect to τ, then it is easy to verify that K has $P(d,\tau)$ with the sequence of sets $A_n = \bigcup \mathfrak{N}_n$. ∎

Corollary 2.5: *If X^* is a dual Banach space and $K \subset X^*$ is a w^*-compact subset having $P(\|.\|, w^*)$, then K is fragmentable by the norm. In particular, if X^* has $P(\|.\|, w^*)$, then X^* has the Radon–Nikodým property.*

The following result compares with [16, Lemma 2.1].

Theorem 2.6: *Let (K_i,τ_i) be compact spaces for $i = 1,2$ and let d_i be metrics on K_i. Suppose that there is a surjection T: $K_1 \to K_2$ such that T is τ_1-τ_2 continuous and d_1-d_2 continuous. If K_1 has $P(d_1,\tau_1)$ with τ_1-closed sets, then K_2 has $P(d_2,\tau_2)$ with τ_2-closed sets.*

Proof: If K_1 has $P(d_1,\tau_1)$ with τ_1-closed sets, there is a τ_1-lsc function F_1: $K_1 \to [0,1]$ with the Kadec property by Proposition 2.2. Define a function F_2: $K_2 \to [0,1]$ as follows:

$$F_2(x) = \inf\{F_1(x'): T(x') = x\}.$$

Since F_1 is τ_1-lsc, this infimum is attained. We claim that F_2 is τ_2-lsc. Indeed, suppose that $\lim_\omega x_\omega = x$ in (K_2,τ_2) and $F_2(x_\omega) \leq r$ for every ω. Take points $x'_\omega \in K_1$ such that $T(x'_\omega) = x_\omega$ and $F_1(x'_\omega) = F_2(x_\omega)$. Let $x' \in K_1$ be a cluster point of (x'_ω). Since F_1 is τ_1-lsc we have that $F_1(x') \leq r$. On the other hand, by continuity, $T(x') = x$, so $F_2(x) \leq F_1(x')$. This shows that $F_2(x) \leq r$ and the claim is proved. We claim now that F_2 has the Kadec property and then the result will follow from Proposition 2.2. Suppose not, that is, there is a net (x_ω) in K_2 with τ_2-limit a point x such that $\lim_\omega F_2(x_\omega) = F_2(x)$, and there is $\varepsilon > 0$ such that $d_2(x_\omega, x) > \varepsilon$. Take points $x'_\omega \in K_1$ such that $T(x'_\omega) = x_\omega$ and

$F_1(x'_\omega) = F_2(x_\omega)$. Let x' be a cluster point of (x'_ω). Without loss of generality we can assume that (x'_ω) is τ_1-converging to x'. Clearly, we have that $T(x') = x$ and the following inequalities,

$$\lim_\omega F_2(x_\omega) = F_2(x) \leq F_1(x') \leq \lim_\omega F_1(x'_\omega) = \lim_\omega F_2(x_\omega).$$

We deduce that $\lim_\omega F_1(x'_\omega) = F_1(x')$. By the Kadec property of F_1 we have that $\lim_\omega d_1(x'_\omega, x') = 0$, and from the d_1-d_2 continuity of T we deduce that $\lim_\omega d_2(x_\omega, x) = 0$, which is a contradiction to our supposition. ∎

COROLLARY 2.7: *Let $T\colon X^* \to Y^*$ be a bounded linear operator between dual spaces which is w^*-w^* continuous. If $K \subset X^*$ is a w^*-compact subset having $P(\|.\|, w^*)$, then $T(K)$ has $P(\|.\|, w^*)$ in Y^*.*

The following result compares with [16, Lemma 2.2].

LEMMA 2.8: *Let (K_n, τ_n) be compact spaces for $i \in \mathbb{N}$ and let d_n be a metric on K_n such that K_n has $P(d_n, \tau_n)$ with τ_n-closed sets for every $n \in \mathbb{N}$. Let τ be the topology product of the τ_n-topologies on $K = \prod_{n=1}^\infty K_n$ and let d be a metric compatible with the product of the d_n-topologies on K. Then K has $P(d, \tau)$ with τ-closed sets.*

Proof: Take for every $n \in \mathbb{N}$ a τ_n-lsc Kadec function $F_n\colon K_n \to [0,1]$. An easy lower semicontinuity argument shows that

$$F(x) = \sum_{n=1}^\infty 2^{-n} F_n(x_n)$$

is a τ-lsc Kadec function on K linking d with τ, where $x = (x_n)$. ∎

The following result can be regarded as a topological version of the transfer technique of Godefroy–Troyanski–Whitfield–Zizler [6, 2]; see Theorem 4.1.

THEOREM 2.9: *Let (X, τ) be a topological space and let d be a τ-lower semicontinuous metric on X. Suppose that there exist τ-compacts sets $K_n \subset X$ having $P(d, \tau)$ such that $\overline{\bigcup_{n=1}^\infty K_n}^d = X$. Then X has $P(d, \tau)$.*

Proof: We can suppose the sequence (K_n) increasing and the metric d bounded. By Proposition 2.2, for every $n \in \mathbb{N}$ there is a τ-lsc Kadec function $F_n\colon K_n \to [0,1]$. We define the functions f_n on X as follows,

$$f_n(x) = \inf\{d(x,y) + F_n(y)\colon y \in K_n\}.$$

Note that the infimum is attained. We claim that f_n is τ-lsc. Indeed, suppose that $\tau\text{-}\lim_\omega x_\omega = x$ and $f_n(x_\omega) \le r$ for every ω. Take points $y_\omega \in K_n$ such that $f_n(x_\omega) = d(x_\omega, y_\omega) + F_n(y_\omega)$. Let $y \in K_n$ be a cluster point of (y_ω). Then we have that

$$f_n(x) \le d(x,y) + F_n(y) \le r$$

because of the lower semicontinuity of d and F_n. Now we define a function F on X by the formula

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x).$$

We claim that F has the Kadec property. Indeed, suppose not. We can take a net (x_ω) in X with τ-limit a point x such that $\lim_\omega F(x_\omega) = F(x)$, and there is $\varepsilon > 0$ such that $d(x_\omega, x) > \varepsilon$. A standard argument of lower semicontinuity gives that $\lim_\omega f_n(x_\omega) = f_n(x)$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ such that $1/n < \varepsilon/3$ and $d(x, K_n) < \varepsilon/3$. We can take points $y_\omega \in K_n$ such that

$$f_n(x_\omega) = d(x_\omega, y_\omega) + F_n(y_\omega).$$

Let $y \in K_n$ be a cluster point of (y_ω). Without loss of generality we can assume that $\tau\text{-}\lim_\omega y_\omega = y$. Since

$$d(x,y) + F_n(y) \le \lim_\omega f_n(x_\omega) = f_n(x) \le d(x,y) + F_n(y)$$

we have that

$$\lim_\omega [d(x_\omega, y_\omega) + F_n(y_\omega)] = d(x,y) + F_n(y).$$

Using the lower semicontinuity, we deduce that $\lim_\omega d(x_\omega, y_\omega) = d(x,y) < \varepsilon/3$ and $\lim_\omega F_n(y_\omega) = F_n(y)$. The last equality gives that $\lim_\omega d(y_\omega, y) = 0$, thus for ω big enough we have that $d(x_\omega, y_\omega) < \varepsilon/3$ and $d(y_\omega, y) < \varepsilon/3$. Since $d(x,y) < \varepsilon/3$ we have that $d(x_\omega, x) < \varepsilon$, which is a contradiction. ∎

Given a Banach space X, a bounded subset $Z \subset X^*$ is said to be norming if there is $\lambda > 0$ such that $\lambda\|x\| \le \sup\{|x^*(x)|\colon x^* \in Z\}$ for all $x \in X$. Notice that the supremun defines an equivalent norm on X which is lower semicontinuous for the topology of convergence on Z, denoted $\sigma(X,Z)$. A linear subspace $Z \subset X^*$ is said to be norming if $B_{X^*} \cap Z$ is a norming subset.

The following result compares with [16, Theorem 2.5].

PROPOSITION 2.10: *Let X be a Banach space, let $Z \subset X^*$ be a norming subset and let $K \subset X^*$ be a bounded $\sigma(X,Z)$-compact subset which has $P(\|.\|, \sigma(X,Z))$. Then $\overline{\text{span}}^{\|\cdot\|}(K)$ and $\overline{\text{aco}}^{\sigma(X,Z)}(K)$ have $P(\|.\|, \sigma(X,Z))$.*

Proof: Let $I(n,m) = [-m,m] \times \cdots \times [-m,m]$ (n times) with the usual topology of $\mathbb{R}^n$. Let $K_{n,m} = I(m,n) \times K \times \cdots \times K$ (n times). If τ is the product topology when K is endowed with $\sigma(X,Z)$, then $K_{n,m}$ is τ-compact. If K is endowed with the norm topology, then the product topology is metrized by a metric that we call d. By Lemma 2.8, $K_{n,m}$ has $P(d,\tau)$. The map $T_{n,m}$ from $K_{n,m}$ to X defined by $T_{n,m}(\alpha_1,\ldots,\alpha_n,x_1,\ldots,x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n$ is clearly τ-$\sigma(X,Z)$ continuous and d-$\|.\|$ continuous, thus every $\sigma(X,Z)$-compact set $T_{n,m}(K_{n,m})$ has $P(\|.\|,\sigma(X,Z))$ by Theorem 2.6. Since $\operatorname{span}(K) = \bigcup_{n,m} T_{n,m}(K_{n,m})$, we have that $\overline{\operatorname{span}}^{\|\cdot\|}(K)$ has $P(\|.\|,\sigma(X,Z))$ by Theorem 2.9. The result for the $\sigma(X,Z)$-closed absolutely convex hull follows from the fact that $\overline{\operatorname{aco}}^{\sigma(X,Z)}(K) = \overline{\operatorname{aco}}^{\|\cdot\|}(K)$ because K is fragmentable by the norm; see [16]. ∎

The following result compares with [16, Theorem 5.6].

THEOREM 2.11: *If K is a Namioka–Phelps compact, then $(B_{C(K)^*}, w^*)$ is also a Namioka–Phelps compact.*

Proof: In the proof of [16, Theorem 5.6] it is shown that if K is a Radon–Nikodým compact, then there is a dual Banach space X^* and a bounded injective w^*-w^*-continuous linear operator $T\colon C(K)^* \to X^*$ such that $T(K)$ is fragmented by the norm $\|.\|$ of X^*. If K is moreover Namioka–Phelps, then $T(K)$ has $P(\|.\|,w^*)$. Then $T(B_{C(K)^*}) = \overline{\operatorname{aco}}^{\|\cdot\|}(T(K))$ has $P(\|.\|,w^*)$ by Proposition 2.10, and thus $T(B_{C(K)^*})$ is Namioka–Phelps. ∎

3. Dual LUR renorming

A dual Banach space X^* having a dual LUR norm has the Radon–Nikodým property. The space $C[0,\omega_1]$ shows that the converse is not true. Fabian and Godefroy proved [FG] that a dual Banach space with the Radon–Nikodým property has an equivalent LUR norm (not necessarily dual, of course). The LUR norm in that case can be made a dual norm under additional hypothesis, e.g., if the predual X is WCD, or the space X^* is itself WCD; see [2]. Following Hansell [7], we say that a dual Banach space X^* is **dual-descriptive** if it has the Radon–Nikodým property and the weak* topology has a σ-isolated network. The class of dual-descriptive spaces coincides with the dual Banach spaces having a countable cover by sets of local small diameter in the sense of Jayne, Namioka and Rogers [10]. A dual Banach space with a w^*-Kadec norm is dual-descriptive [7]. Our main result states that the existence of an equivalent dual LUR norm is a topological property. Partial results in this direction were obtained in [19], in the spirit of the Moltó–Orihuela–Troyanski characterization of LUR renormability [14].

THEOREM 3.1: *Let X^* be a dual Banach space. The following conditions are equivalent:*

(i) *X^* admits an equivalent dual LUR norm.*
(ii) *X^* admits an equivalent norm such that weak topology and the weak* topology coincide on its unit sphere.*
(iii) *X^* is dual-descriptive.*
(iv) *X^* (resp. S_{X^*}) has $P(\|.\|, w^*)$.*

Proof: (i) $\Leftrightarrow$ (ii) It is proved in [19].
(i) $\Rightarrow$ (iv) It follows from Proposition 2.2.
(iv) $\Leftrightarrow$ (iii) It follows from Theorem 2.4.
(iv) $\Rightarrow$ (ii) If a dual Banach space X^* has $P(\|.\|, w^*)$, then X^* has the Radon–Nikodým property, by Corollary 2.5. A result from [18] establishes that there is a w^*-lower semicontinuous real function F on X^* with $\|.\| \leq F(.) \leq 3\|.\|$ such that the norm and the w^*-topology coincide on the set $S = \{x^* \in X^*\colon F(x^*) = 1\}$. Let $K = \{x^* \in X^*\colon F(x^*) \leq 1\}$. Since X^* has the Radon–Nikodým property, $\overline{\mathrm{co}}^{\|\cdot\|}(K)$ will be a w^*-compact set, symmetric and with nonempty norm interior, that is the unit ball of some equivalent dual norm on X^*. Without loss of generality we can suppose X^* endowed with that norm, namely $B_{X^*} = \overline{\mathrm{co}}^{\|\cdot\|}(K)$. We will show that the norm and the w^*-topology coincide on S_{X^*}.

Suppose not, that is, there is some $\varepsilon > 0$ and some net (x^*_ω) in S_{X^*} w^*-converging to a point $x^* \in S_{X^*}$ such that $\|x^*_\omega - x^*\| > \varepsilon$. Take Radon probabilities μ_ω on K such that $x^*_\omega = \int_K \mathbb{I} d\mu_\omega$ (integrals are taken in the sense of Bochner, see [3]). Without loss of generality we can suppose that (μ_ω) converges in $(C(K)^*, w^*)$ to a Radon probability μ on K. We must have that $x^* = \int_K \mathbb{I} d\mu$.

Since $\|x^*_\omega\| = \|x^*\| = 1$, we have that μ_ω and μ are supported by $S_{X^*} \cap K \subset S$. We can take disjoint norm compact sets $K_i \subset S$ for $i = 1, \ldots, n$ with diameter less than $\varepsilon/7$ such that $\mu(\bigcup_{i=1}^n K_i) > 1 - \varepsilon/12$. We can take a norm compact set $K_0 \subset S$ disjoint from $\bigcup_{i=1}^n K_i$ such that $\mu(\bigcup_{i=0}^n K_i) > 1 - \varepsilon/12n$. Take disjoint norm open sets V_i for $i = 0, \ldots, n$ with $K_i \subset V_i$ and the diameter of V_i for $i = 1, \ldots, n$ less that $\varepsilon/6$. Since the norm and the w^*-topology coincide on S, we can take w^*-open sets U_i such that $U_i \cap S = V_i \cap S$.

By Urysohn's Lemma, we can take w^*-continuous functions f_i for $i = 0, \ldots, n$ from B_{X^*} to $[0, 1]$ such that $f_i|_{K_i} = 1$ and $f_i|_{X^* \setminus U_i} = 0$. Since $\int_K f_i d\mu_\omega$ converges to $\int_K f_i d\mu \geq \mu(K_i)$ for $i = 0, \ldots, n$ we will have, for ω big enough, that $\mu_\omega(V_i) = \mu_\omega(U_i) \geq \int_K f_i d\mu_\omega > \mu(K_i) - \varepsilon/12n^2$ for $i = 0, \dots, n$. On the other hand, we must have $\mu_\omega(V_i) < \mu(K_i) + \varepsilon/6n$. If not, then $\mu_\omega(V_j) \geq \mu(K_j) + \varepsilon/6n$ for some j. Summing the above inequalities for $i \neq j$ we will have $\mu_\omega(\bigcup_{i=0}^n V_i) > \mu(\bigcup_{i=0}^n K_i) + \varepsilon/6n - n\varepsilon/12n^2 > 1 - \varepsilon/12n + \varepsilon/6n - \varepsilon/12n = 1$ which is a contradiction. Thus we have that $|\mu_\omega(V_i) - \mu(K_i)| < \varepsilon/6n$ and $\mu_\omega(\bigcup_{i=1}^n V_i) > 1 - \varepsilon/6$.

Fix any $i = 1, \dots, n$. We can take points $x_1^*, x_2^* \in \overline{\mathrm{co}}^{\|\cdot\|}(V_i)$ such that $\mu(K_i)x_1^* = \int_{K_i} \mathbb{I} d\mu$ and $\mu_\omega(V_i)x_2^* = \int_{V_i} \mathbb{I} d\mu_\omega$. Since the diameter of V_i is less than $\varepsilon/6$, then $\|x_1^* - x_2^*\| \leq \varepsilon/6$. We have that

$$\left\| \int_{V_i} \mathbb{I} d\mu_\omega - \int_{K_i} \mathbb{I} d\mu \right\| = \|\mu_\omega(V_i)x_2^* - \mu(K_i)x_1^*\|$$
$$\leq |\mu_\omega(V_i) - \mu(K_i)|.\|x_2^*\| + \mu(K_i)\|x_1^* - x_2^*\| \leq (1/n + \mu(K_i))(\varepsilon/6).$$

We will show that $\|x_\omega^* - x^*\| < \varepsilon$ to get the final contradiction

$$\|x_\omega^* - x^*\| = \left\| \int_K \mathbb{I} d\mu_\omega - \int_K \mathbb{I} d\mu \right\|$$
$$\leq \left\| \int_{K \setminus \bigcup_{i=1}^n V_i} \mathbb{I} d\mu_\omega - \int_{K \setminus \bigcup_{i=1}^n K_i} \mathbb{I} d\mu \right\| + \sum_{i=1}^n \left\| \int_{V_i} \mathbb{I} d\mu_\omega - \int_{K_i} \mathbb{I} d\mu \right\|$$
$$\leq \varepsilon/6 + \varepsilon/12 + \sum_{i=1}^n (1/n + \mu(K_i))(\varepsilon/6) < \varepsilon.$$

This shows that the norm $\|.\|$ is w^*-Kadec. ∎

COROLLARY 3.2: *Let X be a Banach space such that its dual X^* satisfies any of the statements of Theorem 3.1. Then X has an equivalent Fréchet differentiable norm.*

Let X be an Asplund Banach space. We shall consider the following construction on its dual X^*. For any weak*-compact convex subset $B \subset X^*$ and $\varepsilon > 0$, take

$$(B)'_\varepsilon = \{x^* \in B: \forall U \in w^*,\ x^* \in U,\ \mathrm{diam}(B \cap U) > \varepsilon\}.$$

Define by transfinite induction the sets (B_ε^α) as follows:

$$B_\varepsilon^0 = B_{X^*},$$
$$B_\varepsilon^{\alpha+1} = (B_\varepsilon^\alpha)'_\varepsilon$$

and, for α a limit ordinal,

$$B_\varepsilon^\alpha = \bigcap_{\beta<\alpha} B_\varepsilon^\beta.$$

Now take $Sz(X,\varepsilon) = \inf\{\alpha\colon B_\varepsilon^\alpha = \emptyset$, and $Sz(X) = \sup_{\varepsilon>0} \delta_Z(X,\varepsilon)\}$. The ordinal number $Sz(X)$ is called the Szlenk index of X. The following result was proved by Lancien [13] using a Kunen–Martin type argument.

COROLLARY 3.3 (Lancien): *If X is a Banach space with $Sz(X) < \omega_1$, then X^* has an equivalent dual LUR norm.*

4. Transfer and embedding

The following is a transfer result for LUR renorming of Godefroy–Troyanski–Whitfield–Zizler [6, 2]. A topological version of it is Theorem 2.9.

THEOREM 4.1 (Godefroy, Troyanski, Whitfield & Zizler): *Let X be a Banach space, let $Z \subset X^*$ be a norming subset, let Y^* be a dual Banach space having a dual LUR norm and let $T\colon Y^* \to X$ be a bounded linear operator w^*-$\sigma(X,Z)$ continuous. Then X has an equivalent $\sigma(X,Z)$-lower semicontinuous norm which is LUR at the points of $\overline{T(Y^*)}^{\|\cdot\|}$.*

We shall prove the following interpolation result in the spirit of the Davis–Figiel–Johnson–Pelczyński Theorem, that can be regarded as a reciprocal of Theorem 4.1.

THEOREM 4.2: *Let X be a Banach space, let $Z \subset X^*$ be a norming subset and let $K \subset X$ be a bounded $\sigma(X,Z)$-compact subset which has $P(\|.\|,\sigma(X,Z))$. Then there exists a dual Banach space Y^* having a dual LUR norm and a bounded one-to-one linear operator $T\colon Y^* \to X$ which is w^*-$\sigma(X,Z)$ continuous such that $K \subset T(B_{Y^*})$.*

Proof: After Lemma 2.10 we know that $K_0 = \overline{\mathrm{aco}}^{\|\cdot\|}(K)$ is an absolutely convex compact set with $P(\|.\|,\sigma(X,Z))$. Thus K_0 is fragmented by the norm. Following Namioka [16], there is an Asplund space Y and a bounded injective linear operator $T\colon Y^* \to X$ which is w^*-$\sigma(X,Z)$ continuous such that $K_0 \subset T(B_{Y^*}) \subset 2^n K_0 + B[0,1/2^n]$ for every $n \in \mathbb{N}$. By Theorem 2.9 we have that $T(B_{Y^*})$ will be a descriptive $\sigma(X,Z)$-compact subset of X. Since T is a homeomorphism when restricted to B_{Y^*}, we deduce that (B_{Y^*}, w^*) has a σ-isolated network. Thus Y^* is dual-descriptive and it has an equivalent dual LUR norm by Theorem 3.1. ∎

COROLLARY 4.3: *Let X be a Banach space, let $Z \subset X^*$ be a norming subset and let $K \subset X$ be a bounded $\sigma(X,Z)$-compact subset which has $P(\|.\|,\sigma(X,Z))$.*

Then X has an equivalent $\sigma(X,Z)$-lower semicontinuous norm which is LUR at the points of $\overline{\text{span}}^{\|\cdot\|}(K)$.

Proof: Apply Theorems 4.2 and 4.1. ∎

The following extends a well-known result of Deville [1] concerning the dual LUR renorming of $C(K)^*$, where K is a scattered compact space such that $K^{(\omega_1)} = \emptyset$; see also [2].

COROLLARY 4.4: *Let K be a Hausdorff compact space. The following are equivalent:*

(i) *$C(K)^*$ has an equivalent dual LUR norm.*
(ii) *K is a countable union of relatively discrete subsets.*

Proof: Suppose that $C(K)^*$ has an equivalent dual LUR norm. Then $C(K)^*$ has $P(\|.\|, w^*)$ and, in particular, K has $P(\|.\|, w^*)$ with some sequence of subsets (A_n). The following sets

$$D_n = \{x \in A_n : \exists U \in w^*, x \in U, \text{diam}(A_n \cap U) < 1\}$$

are relatively discrete and cover K. Conversely, assume that K is a countable union of relatively discrete subsets. Then it is easy to see that K has $P(d,\tau)$ where d is the discrete metric. By Theorem 2.4, K is d-fragmentable, so K must be scattered. Regarding K as a subset of $C(K)^*$, it has $P(\|.\|, w^*)$ and $C(K)^* = \overline{\text{span}}^{\|\cdot\|}(K)$ has an equivalent dual LUR norm by Corollary 4.3. ∎

Remark 4.5: A Hausdorff compact space K satisfies statement (ii) of Corollary 4.4 if and only if K is scattered and it has a σ-isolated network. There exists a scattered compact space K which is Eberlein, is also a countable union of relatively discrete subsets and $K^{(\omega_1)} \neq \emptyset$; see [2].

PROPOSITION 4.6: *Let K be Hausdorff compact space. The following statements are equivalent:*

(i) *K is Namioka–Phelps.*
(ii) *There is a lower semicontinuous metric d such that K has $P(d,\tau)$.*
(iii) *K is Radon–Nikodým and it has a σ-isolated network.*

Proof: (i) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) It is clear after Theorem 2.4.

(ii) $\Rightarrow$ (i) Let d be a lower semicontinuous metric on (K,τ) such that K has $P(d,\tau)$. There is a dual space X^* containing K as w^*-compact subset in such a way that the metric d is induced by the norm [9]. Then the result will follow from Theorem 4.2. ∎

THEOREM 4.7: *Let K be a Namioka–Phelps compact space. Then $C(K)^*$ has an equivalent W*LUR norm. In particular, $C(K)$ has an equivalent Gâteaux differentiable norm.*

Proof: The proof of [16, Theorem 5.6] shows that if K is a Radon–Nikodým compact, then there is a dual Banach space X^* and a bounded injective w^*-w^*-continuous linear operator $T: C(K)^* \to X^*$ such that $T(K)$ is fragmented by the norm $\|.\|$ of X^*. If K is Namioka–Phelps, we have that $T(K)$ has $P(\|.\|, w^*)$. By Corollary 4.3, we can suppose that X^* is endowed with a dual norm which is LUR at the points of $T(C(K)^*)$. Define an equivalent dual norm $|||.|||$ on $C(K)^*$ by the formula $|||x|||^2 = \|x\|^2 + \|T(x)\|^2$. We claim that $|||.|||$ is W*LUR. To see that, take points x, x_n in $C(K)^*$ with $|||x||| = |||x_n||| = 1$ and $\lim_n |||x_n + x||| = 2$. By a standard convexity argument [2, p. 45], we have that $\lim_n \|T(x_n)\| = \|T(x)\|$ and $\lim_n \|T(x_n) + T(x)\| = 2\|T(x)\|$. Since $\|.\|$ is LUR at $T(x)$, we have that $\lim_n \|T(x_n) - T(x)\| = 0$. In particular, $T(x_n)$ is w^*-convergent to $T(x)$, and hence (x_n) is w^*-convergent to x because of the w^*-continuity of T^{-1} on $T(B_{C(K)^*})$. ∎

Remark 4.8: If K is a scattered Namioka–Phelps compact space, then it verifies the hypothesis of Corollary 4.4. In particular, $C(K)$ has an equivalent Fréchet differentiable norm.

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